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WAVENUMBER EXPLICIT ANALYSIS FOR GALERKIN DISCRETIZATIONS OF LOSSY HELMHOLTZ PROBLEMS*

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Abstract. We present a stability and convergence theory for the lossy Helmholtz equation and its Galerkin discretization. The boundary conditions are of Robin type. All estimates are explicit with respect to the real and imaginary parts of the complex wavenumber $\zeta \in \mathbb{C}$, $\operatorname{Re} \zeta \geq 0$, $|\zeta| \geq 1$. For the extreme cases $\zeta \in \mathrm{i}\mathbb{R}$ and $\zeta \in \mathbb{R}_{\geq 0}$, the estimates coincide with the existing estimates in the literature and exhibit a seamless transition between these cases in the right complex half plane.

Key words. Helmholtz equation, stability, *hp*-finite elements

AMS subject classifications. 35J05, 65N30, 65N12

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1. Introduction.

The understanding and simulation of phenomena related to wave propagation and scattering have numerous applications in science and engineering. *Time harmonic* wave phenomena with *loss* and *absorption* are widely used models for many applications, such as, e.g., in viscoelastodynamics for materials with damping (see, e.g., [1]), in electromagnetics for wave propagation in lossy media (see, e.g., [10]), and in nonlinear optics (see, e.g., [18]). In the simplest case such problems are modeled by a *Helmholtz* equation with complex wavenumber. For homogeneous, isotropic material the differential operator is given by

$$\mathcal{L}_\zeta u := -\Delta u + \zeta^2 u,$$

where $\zeta = \operatorname{Re} \zeta + \mathrm{i} \operatorname{Im} \zeta =: \nu - \mathrm{i}\, k$ with $\nu \geq 0$ and $k \in \mathbb{R}$ denotes the wavenumber. For $\operatorname{Re} \zeta = 0$, the operator corresponds to the *pure Helmholtz problem*, for $\operatorname{Im} \zeta = 0$ the corresponding equation is called the *modified Helmholtz equation* or *screened Poisson problem*. If one restricts ζ to a conical sector, i.e., $|k| \leq \beta\nu$ for some fixed positive β, we call it the "sectorial case". Besides the modeling of wave propagation in lossy media this operator appears also in other applications such as *convolution quadrature* for the wave equation or the limiting absorbing principle in the context of fast solvers for Helmholtz problems. Such applications are sketched in more detail, e.g., in [3].

Another important application is the approximation of the inverse Laplace transform by contour quadrature where the Helmholtz operator has to be discretized at many complex frequencies in the right complex half plane (see, e.g., [11]).

For the two extreme cases $\zeta = -\mathrm{i}\, k$ and $\zeta = \nu$, $k \in \mathbb{R}$, $\nu \in \mathbb{R}_{\geq 0}$, a fairly complete theory for standard Galerkin *hp*-finite element methods (*hp*-FEM) is available, and the error estimates are explicit with respect to the wavenumber ζ, the mesh width h of the finite element mesh, and the polynomial degree p: (a) For $\zeta = -\mathrm{i}\, k$ and large

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$|k|$ the problem is highly indefinite. We define for any $c, C > 0$ and large $|k| > 0$ a "resolution condition" for h and p of the form

$$R(c, C, k): \qquad \frac{|k|h}{p} \leq c \quad \wedge \quad p \geq C \log |k| . \tag{1.1}$$

The results in [6, 13, 14, 15, 16] imply the following: For any fixed $C > 0$, there exists $c > 0$ such that the resolution condition $R(c, C, k)$ implies solvability of the *hp*-FEM Galerkin discretization and quasi-optimality. (b) For $\zeta = \nu > 0$ and $\nu = O(1)$, the problem is properly elliptic and Céa's lemma ensures well-posedness and quasi-optimality without any resolution condition. (c) For $\zeta = \nu \gg 1$, the solution exhibits boundary layers. Although the Galerkin discretization is always well-posed in case (c), special meshes should be used that are adapted to the boundary layers (see, e.g., [12, 17, 23] and references therein). In this paper, we will develop a unified theory for Galerkin discretizations of $\mathcal{L}_\zeta$ with Robin boundary conditions that is applicable for all $\zeta \in \mathbb{C}$, $\operatorname{Re}\zeta \geq 0$, and $|\zeta| \geq 1$. All estimates are explicit in terms of $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ and reproduce the limiting cases of purely real and imaginary ζ. It is shown that, for the *sectorial case*, i.e., the wavenumber lies in a sectorial neighborhood of the real axis in the right complex half plane, well-posedness and quasi-optimality are a consequence of coercivity, while for $\operatorname{Re}\zeta \to 0$ the estimates tend continuously to the purely imaginary case $\zeta = -\mathrm{i}\,k$. We follow the general theory developed in [15, 16] and refine the estimates to be explicit with respect to the real and imaginary parts of the wavenumber.

The paper is structured as follows. In section 2 we introduce the Helmholtz model problem with Robin boundary conditions and formulate some geometric and algebraic assumptions on the data. Further, we define for the wavenumber the (well-behaved) sectorial and the (more critical) nonsectorial region.

The estimate of the continuity constant for the sesquilinear form is derived in section 3. Section 4 is devoted to the analysis of the inf-sup constant for the continuous sesquilinear form. If the real part of the wavenumber is positive, the estimate follows simply from the coercivity of the sesquilinear form. However, this bound degenerates as $\operatorname{Re}\zeta \to 0$. This can be remedied by a different proof: first one uses suitable test functions to derive stability estimates for an adjoint problem with L^2 right-hand sides and then employs this result for the estimate of the inf-sup constant in the vicinity of the imaginary axis.

The key role for the analysis of the Galerkin discretization is played by a regular decomposition of the Helmholtz solution. In section 5, we introduce a splitting of the Helmholtz solution into a part with (low) H^2-regularity and wavenumber-*independent* regularity constant and an analytic part with a more critical wavenumber dependence. First, this is derived for the full space solution by generalizing the results for purely imaginary frequencies in [15]. In the case of bounded domains, we generalize the *iteration argument* in [16, sect. 4] to general complex frequencies. In addition, this requires sharp estimates of frequency-dependent lifting operators which we also present in this section.

Section 6 is devoted to the estimate of the discrete inf-sup constant for the standard Galerkin discretization of the Helmholtz equation. We will derive two types of estimates: one requires that the finite dimensional space for the Galerkin discretization satisfies a certain *resolution condition* and allows for robust (as $\operatorname{Re}\zeta \to 0$) stability and quasi-optimal convergence estimates; the other estimate avoids a resolution condition while the constants in the estimates tend towards ∞ as $\operatorname{Re}\zeta \to 0$ but stay robust for the *sectorial case*. Numerical examples in section 7 illustrate the application of our analysis in the context of the *hp*-finite element method (*hp*-FEM).

In summary, the main achievements in this paper are as follows.

(a) Inf-sup constant. We prove an estimate of the inf-sup constant γ_ζ for the lossy Helmholtz problem with impedance boundary conditions which is explicit in the real and imaginary parts of the frequency

$$\gamma_\zeta \geq \frac{1}{1 + c\frac{|\operatorname{Im}\zeta|}{1+\operatorname{Re}\zeta}}.$$

For conforming Galerkin discretizations, estimates of the *discrete* inf-sup constant γ_{disc} are derived in Theorem 6.2 for different *resolution conditions*, e.g.,

$$\frac{(\operatorname{Im}\zeta)^2}{|\zeta|}\eta\,(S) \leq C \implies \gamma_{\mathrm{disc}} \geq c\frac{1+\operatorname{Re}\zeta}{|\zeta|}$$

for sufficiently small constant C. Here, $\eta\,(S)$ is an *adjoint approximation property* which will be defined in section 6.

(b) Regular decomposition. We generalize the splitting lemma in [15] to derive a regular decomposition of the full space solution which allows for stability and regularity estimates which are explicit in $\operatorname{Im}\zeta$ and $\operatorname{Re}\zeta$. The proof requires more subtle estimates of the symbol for the full space operator compared to [15].

The derivation of a regular decomposition for the solution of the lossy Helmholtz problem on a bounded domain is also based on an iteration argument, where the Helmholtz operator with the "good" sign has to be analyzed. It turns out that more subtle trace liftings and sharper estimates are needed compared to those in [16, sect. 4] in order to get optimal dependencies on $\operatorname{Im}\zeta$ and $\operatorname{Re}\zeta$ (see Lemma 5.4).

(c) Application to *hp*-FEM. The most prominent application of our theory is the Galerkin *hp*-finite element method. We use *hp*-interpolation estimates and combine them with the stability and abstract convergence analysis to derive resolution conditions and quasi-optimal convergence estimates which are explicit in h, p, $\operatorname{Re}\zeta$, and $\operatorname{Im}\zeta$. Numerical results illustrate the sharpness of the analysis.

2. Setting. Throughout this paper, we assume $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. As our model problem we consider the Helmholtz problem with impedance boundary conditions imposed on the whole boundary

$$\begin{aligned} -\Delta u + \zeta^2 u &= f \qquad \text{in } \Omega, \\ \partial_n u + \zeta u &= g \qquad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{2.1}$$

for $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$. We assume that the wavenumber (frequency) ζ satisfies[1]

$$\zeta \in \mathbb{C}^\circ_{\geq 0} := \{\zeta \in \mathbb{C}_{\geq 0} \mid |\zeta| \geq 1\}, \tag{2.2}$$

where, for $\rho \in \mathbb{R}$,

$$\mathbb{C}_{>\rho} := \{\xi \in \mathbb{C} \mid \operatorname{Re}\xi > \rho\} \quad \text{and} \quad \mathbb{C}_{\geq\rho} := \{\xi \in \mathbb{C} \mid \operatorname{Re}\xi \geq \rho\}.$$

Remark 2.1. We have chosen (2.1) as our model problem for the following reasons. The generalization of the theory in [15, 16] to (2.1) which is explicit in $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$

[1]The condition $|\zeta| \geq 1$ can be replaced by $|\zeta| \geq \rho_0$ for any $\rho_0 > 0$. However, the constants in our estimates possibly deteriorate as $\rho_0 \to 0$.

contains practically all theoretical difficulties which arise for the following related problems:

Lossy Helmholtz equations with transparent boundary conditions which correspond to a full space problem are simpler than the one considered here since only the acoustic Newton potential is involved. This operator is analyzed in our paper as well.

We have excluded for our frequencies a ball with radius 1 about the origin from $\mathbb{C}_{\geq 0}$ because problem (2.1) becomes ill-posed as $\zeta \in \mathbb{C}_{\geq 0}$ tends to 0, i.e., to the Neumann problem of the Poisson equation. However, this low-frequency case is studied in detail in the literature on elliptic problems, and so we have omitted this in our paper.

Dirichlet boundary conditions also result in an ill-posed problem for certain purely imaginary frequencies. In [16] an exterior Dirichlet problem on the complement of a bounded domain with an analytic boundary has been considered. For this, one has to employ graded meshes, but we decided not to include this case here in order to avoid too many technicalities.

Note that the choice $\zeta = -\,\mathrm{i}\,k$ leads to the standard Helmholtz case. The frequency domain $\mathbb{C}_{\geq 0}^{\circ}$ is split into the *sectorial* and *nonsectorial* domains

$$D_\beta := \{\xi \in \mathbb{C}_{\geq 0}^{\circ} : |\operatorname{Im} \xi| < \beta \operatorname{Re} \xi\}, \qquad D_\beta^c := \{\xi \in \mathbb{C}_{\geq 0}^{\circ} : |\operatorname{Im} \xi| \geq \beta \operatorname{Re} \xi\}$$

for some $\beta > 0$.

Our focus is on the derivation of stability and error estimates that are explicit in the real and imaginary parts of ζ but less on the development of a theory with minimal assumptions on the geometry of the domain. In this light, we define the following class of domains denoted by $\mathcal{A}$.

Definition 2.2. *A domain $\Omega \subset \mathbb{R}^3$ belongs to $\mathcal{A}$ if Ω is a bounded Lipschitz domain with analytic boundary that is star-shaped with respect to a ball.*

We note that our results can be extended to convex polygonal domains by following the arguments developed in [16].

Let $L^2(\Omega)$ denote the usual Lebesgue space with scalar product denoted by $(\cdot,\cdot)$ (complex conjugation is on the second argument) and norm $\|\cdot\|_{L^2(\Omega)} := \|\cdot\| := (\cdot,\cdot)^{1/2}$. Let $V = H^1(\Omega)$ denote the usual Sobolev space and let $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ be the standard trace operator. We introduce the sesquilinear forms

$$a_{0,\zeta}(u,v) := (\nabla u, \nabla v) + \left(\zeta^2 u, v\right) \qquad \forall u, v \in V,$$

and

$$b_\zeta(\gamma_0 u, \gamma_0 v) := (\zeta \gamma_0 u, \gamma_0 v)_\Gamma \qquad \forall u, v \in V,$$

where $(\cdot,\cdot)_\Gamma$ is the $L^2(\Gamma)$ scalar product.

The weak formulation of the Helmholtz problem with Robin boundary conditions (2.1) is given as follows: For $F = (f, \cdot) + (g, \gamma_0 \cdot)_\Gamma \in V'$, we seek $u \in V$ such that

$$a_\zeta(u,v) := a_{0,\zeta}(u,v) + b_\zeta(\gamma_0 u, \gamma_0 v) = F(v) \quad \forall v \in V. \tag{2.3}$$

In the following, we will write $(u,v)_\Gamma$ as shorthand for $(\gamma_0 u, \gamma_0 v)_\Gamma$.

3. The continuity constant. In this section, we will estimate the continuity constant of the sesquilinear form $a_\zeta(\cdot,\cdot)$. We equip the Sobolev space V with the indexed norm $\|\cdot\|_{|\zeta|}$, where, for $\rho > 0$, we set

$$\|u\|_{\rho,\Omega} = \|u\|_\rho := \left(\|\nabla u\|^2 + \rho^2 \|u\|^2\right)^{1/2}. \tag{3.1}$$

More generally, for measurable subsets $T \subset \Omega$ we write

$$\|u\|_{\rho,T} := \left(\|\nabla u\|^2_{L^2(T)} + \rho^2 \|u\|^2_{L^2(T)}\right)^{1/2}.$$

The L^2-norm on Γ is denoted by $\|\cdot\|_\Gamma$. On $H^{1/2}(\Gamma)$ we introduce the weighted norm

$$\|g\|_{\Gamma,\rho} := \left(\|g\|^2_{H^{1/2}(\Gamma)} + \rho\|g\|^2_\Gamma\right)^{1/2} \tag{3.2}$$

for $\rho > 0$.

THEOREM 3.1. *The sesquilinear form a_ζ is continuous, and*

$$|a_\zeta(u,v)| \leq (1 + C_b)\|u\|_{|\zeta|}\|v\|_{|\zeta|} \quad \forall u, v \in H^1(\Omega) \tag{3.3}$$

with C_b independent of $\zeta \in \mathbb{C}_{\geq 0}$.

Proof. The continuity estimate for the sesquilinear form $b_\zeta(\cdot,\cdot)$ is a simple consequence of the multiplicative trace inequality (see, e.g., [9, p. 41, last formula])

$$\|\gamma_0 u\|_\Gamma \leq C_{\text{trace}} \|u\|^{1/2} \|u\|^{1/2}_{H^1(\Omega)}. \tag{3.4}$$

Hence,

$$\sqrt{|\zeta|}\,\|\gamma_0 u\|_{L^2(\Gamma)} \leq C_{\text{trace}} (|\zeta|\,\|u\|)^{1/2} \|u\|^{1/2}_{H^1(\Omega)} \leq C\|u\|_{|\zeta|}, \tag{3.5}$$

which implies the continuity of $b_\zeta(\cdot,\cdot)$,

$$|b_\zeta(\gamma_0 u, \gamma_0 v)| \leq C_b \|u\|_{|\zeta|}\|v\|_{|\zeta|} \qquad \forall u, v \in H^1(\Omega), \tag{3.6}$$

for a constant C_b independent of $\zeta \in \mathbb{C}^\circ_{\geq 0}$ and u, v. ∎

4. The inf-sup constant of $a_\zeta(\cdot,\cdot)$. Our goal in this section is to estimate the inf-sup constant

$$\gamma_\zeta := \inf_{u\in V}\sup_{v\in V}\frac{|a_\zeta(u,v)|}{\|u\|_{|\zeta|}\|v\|_{|\zeta|}}, \tag{4.1}$$

which implies well-posedness of (2.3). This involves two different theoretical techniques: In section 4.1 we consider the case $\operatorname{Re}\zeta > 0$ and obtain estimates from the coercivity of the sesquilinear form. These estimates give stable bounds for the sectorial case but deteriorate as $\operatorname{Re}\zeta \to 0$ in the nonsectorial case. In section 4.2 we employ the sesquilinear form with a suitably selected test function and obtain sharp estimates also for the nonsectorial case.

4.1. The inf-sup constant for $\operatorname{Re}\zeta > 0$. The estimate of the inf-sup constant in Lemma 4.1 is a direct consequence of the technique used in [2].

LEMMA 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and let $\zeta \in \mathbb{C}^\circ_{>0}$. Then the inf-sup constant γ_ζ of* (4.1) *for the sesquilinear form $a_\zeta(\cdot,\cdot)$ (cf.* (2.3)*) satisfies*

$$\gamma_\zeta \geq \frac{\operatorname{Re}\zeta}{|\zeta|}. \tag{4.2}$$

For every $F \in V'$, problem (2.3) *has a unique solution. In particular, if there are $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ such that $F(v) = (f,v) + (g,v)_\Gamma$, then the solution u satisfies*

$$\|u\|_{|\zeta|} \leq \frac{1}{\operatorname{Re}\zeta}\left(\|f\| + C\sqrt{|\zeta|}\,\|g\|_\Gamma\right). \tag{4.3}$$

Proof. We follow the idea of the proof in [2]. We choose $v = \frac{\zeta}{|\zeta|}u$. For the sesquilinear form with Robin boundary conditions we have

$$\operatorname{Re} a_\zeta\left(u, \frac{\zeta}{|\zeta|}u\right) = \frac{\operatorname{Re}\zeta}{|\zeta|}\|u\|_{|\zeta|}^2 + |\zeta|\,\|u\|_\Gamma^2 \geq \frac{\operatorname{Re}\zeta}{|\zeta|}\|u\|_{|\zeta|}^2 .$$

The positivity of the inf-sup constant γ_ζ implies unique solvability (see, e.g., [19, Thm. 2.1.44]; the above argument can be used to show [19, (2.34b)]). We obtain

$$\|u\|_{|\zeta|} \leq \frac{|\zeta|}{\operatorname{Re}\zeta} \sup_{v\in H^1(\Omega)\backslash\{0\}} \frac{|F(v)|}{\|v\|_{|\zeta|}} \leq \frac{|\zeta|}{\operatorname{Re}\zeta}\left(\frac{\|f\|}{|\zeta|} + \|g\|_{L^2(\Gamma)} \sup_{v\in H^1(\Omega)\backslash\{0\}} \frac{\|v\|_\Gamma}{\|v\|_{|\zeta|}}\right).$$

A multiplicative trace inequality in the form of (3.5) leads to (4.3). ◻

Before we can prove an estimate of the inf-sup constant, we need a preparatory lemma which gives an estimate in the case of L^2 data.

LEMMA 4.2. *Let $\Omega \in \mathcal{A}$. Let the functional $F \in V'$ be of the form $F(v) = (f, v) + (g, v)_\Gamma$ with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$. Then, problem* (2.3) *has a unique solution and satisfies*

$$\|u\|_{|\zeta|} \leq C_{\text{stab}}\left(\frac{1}{1+\operatorname{Re}(\zeta)}\|f\| + \frac{1}{\sqrt{1+\operatorname{Re}(\zeta)}}\|g\|_\Gamma\right) \tag{4.4}$$

for some C_{stab} independent of $\zeta \in \mathbb{C}^\circ_{\geq 0}$.

Proof. We distinguish between two cases.

Case a: $\zeta \in D_\beta$. The condition $|\zeta| \geq 1$ leads to

$$\operatorname{Re}\zeta > \left(1+\beta^2\right)^{-1/2}|\zeta| \geq \left(1+\beta^2\right)^{-1/2}, \tag{4.5}$$

and Lemma 4.1 becomes applicable:

$$\gamma_\zeta \geq \frac{\operatorname{Re}(\zeta)}{|\zeta|} \geq \frac{1}{\sqrt{1+\beta^2}},$$

which implies (4.4) for $\zeta \in D_\beta$.

Case b: $\zeta \in D_\beta^{\mathrm{c}}$. For $k > 0$, the lemma follows from [8, Corollary 2.11]. An inspection of the proof shows that the case $k < 0$ follows in the same way. We note that, in this case, the parameter ν relates to the parameter ε in [8] as $\nu \sim \frac{|\varepsilon|}{|k|}$. This means that $\nu \lesssim |k|$ corresponds to the case $|\varepsilon| \lesssim k^2$. ◻

4.2. The inf-sup constant of $a_\zeta(\cdot,\cdot)$ for $\zeta \in D_\beta^{\mathrm{c}}$. In Theorem 4.3 we will prove an alternative estimate (compared to (4.2)) for the inf-sup constant that is robust as $\operatorname{Re}\zeta \to 0$. To estimate this constant we employ the standard ansatz $u \in V$ and $v = u + z$ for some $z \in V$. Then

$$a_\zeta(u, u+z) = \|u\|_{|\zeta|}^2 + a_\zeta(u, z) + b_\zeta(\gamma_0 u, \gamma_0 u) + \left(\zeta^2 - |\zeta|^2\right)\|u\|^2 .$$

The choice of z will be related to some adjoint problem in the next section.

THEOREM 4.3. *Let $\Omega \in \mathcal{A}$. Then there exists a constant $c > 0$ such that for all $\zeta \in \mathbb{C}^\circ_{\geq 0}$ the inf-sup constant γ_ζ of* (4.1) *satisfies*

$$\gamma_\zeta \geq \frac{1}{1 + c\frac{|\operatorname{Im}\zeta|}{1+\operatorname{Re}\zeta}}.$$

Proof. Let $\nu = \operatorname{Re}\zeta$ and $k = -\operatorname{Im}\zeta$, and set $\sigma = 1/\sqrt{2}$. First, we consider the case $\zeta \in \mathbb{C}^{\circ}_{\geq 0}$ with $\nu \geq \sigma$.

From Lemma 4.1 we have for any $\zeta \in \mathbb{C}^{\circ}_{\geq \sigma}$ the estimate

$$\gamma_\zeta \geq \frac{\operatorname{Re}\zeta}{|\zeta|} = \frac{1}{\sqrt{1+\left(\frac{k}{\nu}\right)^2}} \geq \frac{1}{1+\frac{|k|}{\nu}} \geq \frac{1}{1+c\frac{|k|}{\nu+1}} \quad \text{for } c = 1+\sqrt{2}.$$

It remains to consider the case $\zeta \in \mathbb{C}^{\circ}_{\geq 0}$ with $\nu < \sigma$. Let $u,\ z \in V$ and set $v = u + z$. Then

$$a_\zeta(u,v) = \|u\|^2_{|\zeta|} + \left(\zeta^2 - |\zeta|^2\right)\|u\|^2 + \zeta (u,u)_\Gamma + a_\zeta(u,z). \tag{4.6}$$

We consider the adjoint problem: find $z \in V$ such that

$$a_{\overline{\zeta}}(z,w) = \alpha^2(u,w) \quad \forall w \in V \quad \text{with} \quad \alpha^2 := |\zeta|^2 - \overline{\zeta}^2 = -2k\,\mathrm{i}\,\overline{\zeta}, \tag{4.7}$$

which is well-posed according to Lemma 4.2 and satisfies

$$\|z\|_{|\zeta|} \leq C_{\mathrm{stab}}|\alpha|^2\|u\| = 2C_{\mathrm{stab}}|k\zeta|\,\|u\| \leq 2C_{\mathrm{stab}}|k|\,\|u\|_{|\zeta|}.$$

For this choice of z, we consider the real part of (4.6) and obtain

$$\operatorname{Re} a_\zeta(u,v) \geq \|u\|^2_{|\zeta|} + \nu\|u\|^2_\Gamma \geq \|u\|^2_{|\zeta|}.$$

Hence, $\|v\|_{|\zeta|} \leq (1 + 2C_{\mathrm{stab}}|k|)\|u\|_{|\zeta|}$ and

$$\gamma_\zeta \geq \frac{1}{1+2C_{\mathrm{stab}}|k|} \geq \frac{1}{1+\tilde{c}\frac{|k|}{\nu+1}} \quad \text{for } 0 \leq \nu \leq \sigma. \qquad \square$$

5. Regular decomposition of the Helmholtz solution. In this section, we develop a regular decomposition of the solution of the Helmholtz problem (2.1) based on a frequency splitting of the right-hand side. For functions defined on the full space $\mathbb{R}^3$ and for $\zeta \in \mathbb{C}^{\circ}_{\geq 0}$ with $\operatorname{Im}\zeta \neq 0$, the frequency splitting is defined via their Fourier transform (section 5.1, Lemma 5.1). The result of Lemma 5.1 is needed in sections 5.2 and 5.3 for $\zeta \in D^c_\beta$ (cf. Remark 5.2) and yields a regular splitting using a lifting operator for functions defined on finite domains. This generalizes the theory developed in [15, 16] to complex frequencies, and the resulting estimates are explicit with respect to the real and imaginary parts of the wavenumber.

5.1. The full space adjoint problem. The first result concerns the adjoint problem for the full space $\Omega = \mathbb{R}^3$. Let $\phi \in L^2(\Omega)$ be a function with compact support. We choose $R > 0$ sufficiently large so that the open ball B_R with radius R centered at the origin contains $\operatorname{supp}\phi$. We consider the problem

$$\begin{aligned} &(-\Delta + \overline{\zeta^2})z = \phi && \text{in } \mathbb{R}^3, \\ &\left|\left\langle \frac{x}{\|x\|}, \nabla z(x)\right\rangle + \overline{\zeta} z(x)\right| = o\left(\|x\|^{-1}\right) && \text{as } \|x\| \to \infty. \end{aligned} \tag{5.1}$$

To analyze this equation we employ the Fourier transformation and introduce a cutoff function $\mu \in C^\infty(\mathbb{R}_{\geq 0})$ satisfying

$$\begin{aligned} &\operatorname{supp}\mu \subset [0,4R], && \mu|_{[0,2R]} = 1, \quad |\mu|_{W^{1,\infty}(\mathbb{R}_{\geq 0})} \leq \frac{C_\mu}{R}, \\ &\forall x \in \mathbb{R}_{\geq 0} : 0 \leq \mu(x) \leq 1, && \mu|_{[4R,\infty[} = 0, \quad |\mu|_{W^{2,\infty}(\mathbb{R}_{\geq 0})} \leq \frac{C_\mu}{R^2}. \end{aligned} \tag{5.2}$$

The fundamental solution to the Helmholtz operator $\mathcal{L}_\zeta u = -\Delta u + \zeta^2 u$ in $\mathbb{R}^3$ is given by[2]

$$K(\zeta, x) := \kappa(\zeta, \|x\|) \quad \text{with} \quad \kappa(\zeta, r) := (4\pi r)^{-1} \mathrm{e}^{-\zeta r}.$$

It satisfies

$$\left| \left\langle \frac{x}{\|x\|}, \nabla_x K(\zeta, x) \right\rangle + \zeta K(\zeta, x) \right| = o\left(\|x\|^{-1}\right) \quad \text{for } \|x\| \to \infty$$

so that z is given by $z = K(\overline{\zeta}) * \phi$. Define $M(x) := \mu(\|x\|)$ and

$$z_\mu(x) := \left(K\left(\overline{\zeta}\right) M\right) * \phi := \int_{B_R} K\left(\overline{\zeta}, x-y\right) M(x-y)\, \phi(y)\, dy \qquad \forall x \in \mathbb{R}^3.$$

The properties of μ ensure $z_\mu|_{B_R} = z|_{B_R}$. To analyze the stability and regularity of z_μ we introduce a frequency splitting of the solution $z_\mu = z_{H^2} + z_{\mathcal{A}}$ that depends on the complex frequency $\zeta \in \mathbb{C}_{\geq 0}$ and a parameter $\lambda \geq \lambda_0 > 1$.

Lemma 5.1. *Let $\phi \in L^2(\mathbb{R}^3)$ such that* $\operatorname{supp}\phi$ *is contained in a ball $B_R := B_R(0)$ of radius $R > 0$ centered at the origin, and let μ be a cutoff function satisfying* (5.2). *Then there exists a constant $C > 0$ depending only on R and μ such that the solution $z = K(\overline{\zeta}) * \phi$ of* (5.1) *and $z_\mu := \left(K\left(\overline{\zeta}\right) M\right) * \phi$ satisfy $z|_{B_R} = z_\mu|_{B_R}$ and*

$$\|z_\mu\|_{|\zeta|} \leq \frac{C}{1 + \operatorname{Re}\zeta} \|\phi\| \qquad \forall \zeta \in \mathbb{C}_{\geq 0}. \tag{5.3}$$

Furthermore, for every $\lambda \geq \lambda_0 > 1$ and $\zeta \in \mathbb{C}_{\geq 0}$ with $\operatorname{Im}\zeta \neq 0$ there exists a λ- and ζ-dependent splitting $z_\mu = z_{H^2} + z_{\mathcal{A}}$ satisfying

$$\|\nabla^p z_{H^2}\| \leq C' \frac{\lambda}{\lambda - 1} \left(\frac{|\zeta|}{\operatorname{Im}\zeta}\right)^2 (\lambda |\operatorname{Im}\zeta|)^{p-2} \|\phi\| \qquad \forall p \in \{0, 1, 2\}, \tag{5.4}$$

$$\|\nabla^p z_{\mathcal{A}}\| \leq C' \frac{1 + |\zeta|}{1 + \operatorname{Re}\zeta} \left(\sqrt{3}\lambda |\operatorname{Im}\zeta|\right)^{p-2} \|\phi\| \qquad \forall p \in \mathbb{N}_0,\ p \geq 2. \tag{5.5}$$

Here, $|\nabla^p z_{\mathcal{A}}|$ stands for a sum over all derivatives of order p (see (5.13)*). The constant C' depends only on λ_0, R, and μ.*

Remark 5.2. As the estimates in Lemma 5.1 degenerate for $\operatorname{Im}\zeta \to 0$, we will employ Lemma 5.1 for $\zeta \in D_\beta^c$, for fixed $\beta > 0$. Then $|\operatorname{Im}\zeta| \geq \beta \operatorname{Re}\zeta$, and we have

$$|\operatorname{Im}\zeta| \leq |\zeta| \leq \tilde{C} |\operatorname{Im}\zeta| \quad \text{with} \quad \tilde{C} := \frac{\sqrt{1+\beta^2}}{\beta}. \tag{5.6}$$

In particular, $\zeta \in D_\beta^c$ implies $\operatorname{Im}\zeta \neq 0$.

Proof. For $\zeta \in \mathbb{C}_{\geq 0}$, we set $\nu = \operatorname{Re}\zeta$ and $k = -\operatorname{Im}\zeta$. In order to construct the splitting $z = z_{H^2} + z_{\mathcal{A}}$, we start by recalling the definition of the Fourier transformation for functions with compact support

$$\hat{w}(\xi) = \mathcal{F}(w) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle \xi, x\rangle} w(x)\, dx \qquad \forall \xi \in \mathbb{R}^d$$

[2]For $\zeta \in \mathbb{R}_{>0}$, the fundamental solution is denoted as the "Yukawa" potential.

and the inversion formula

$$w(x) = \mathcal{F}^{-1}(w) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} \hat{w}(\xi)\, d\xi \qquad \forall x \in \mathbb{R}^d.$$

Next, we introduce a frequency splitting of a function $w \in L^2(\Omega)$ depending on $\zeta = \nu - \mathrm{i}\, k$ and a parameter $\lambda > 1$ by using the Fourier transformation. The low- and high-frequency parts of w are given by

$$L_{\mathbb{R}^3} w := \mathcal{F}^{-1}\left(\chi_{\lambda|k|} \mathcal{F}(w)\right) \quad \text{and} \quad H_{\mathbb{R}^3} w := \mathcal{F}^{-1}\left(\left(1 - \chi_{\lambda|k|}\right) \mathcal{F}(w)\right), \tag{5.7}$$

where χ_δ is the characteristic function of the open ball with radius $\delta > 0$ centered at the origin.

We construct a decomposition of $z_\mu = z_{H^2} + z_{\mathcal{A}}$ as follows: We decompose the right-hand side ϕ in (5.1) via

$$\phi = \phi_{|k|} + \phi_{|k|}^c = L_{\mathbb{R}^3}\phi + H_{\mathbb{R}^3}\phi. \tag{5.8}$$

Accordingly, we define the decomposition of z_μ by

$$z_{H^2} := \left(K\left(\overline{\zeta}\right) M\right) \star \phi_{|k|}^c \quad \text{and} \quad z_{\mathcal{A}} := \left(K\left(\overline{\zeta}\right) M\right) \star \phi_{|k|}. \tag{5.9}$$

The Fourier transform of $K\left(\overline{\zeta}, \cdot\right) M$ is given by

$$\left(K\widehat{\left(\overline{\zeta}, \cdot\right)} M\right)(\xi) = \sigma\left(\overline{\zeta}, \|\xi\|\right)$$

with symbol

$$\sigma(\zeta, s) = (2\pi)^{-3/2} 4\pi \int_0^\infty \kappa(\zeta, r)\, \mu(r)\, r^2 \frac{\sin(rs)}{rs} dr.$$

In the following we will analyze the symbol $\sigma(\zeta, \cdot)$. We have

$$\begin{aligned}
|s\sigma(\zeta, s)| &= (2\pi)^{-3/2} \left| \int_0^\infty \mathrm{e}^{-\zeta r} \mu(r) \sin(rs)\, dr \right| \\
&\leq (2\pi)^{-3/2} \int_0^{4R} \mathrm{e}^{-\nu r}\, dr = (2\pi)^{-3/2} 4R E_0(4R\nu)
\end{aligned}$$

with $E_0(t) := \frac{1-\mathrm{e}^{-t}}{t} \leq \frac{C_0}{1+t}$. Applying integration by parts leads to

$$\begin{aligned}
\sigma(\zeta, s) &= (2\pi)^{-3/2} \frac{1}{\zeta} \int_0^\infty \mathrm{e}^{-\zeta r}\, \partial_r \left(\mu(r) \frac{\sin(rs)}{s} \right) dr \\
&= (2\pi)^{-3/2} \frac{1}{\zeta} \int_0^\infty \mathrm{e}^{-\zeta r} \left(\mu'(r) \frac{\sin(rs)}{s} + \mu(r) \cos rs \right) dr.
\end{aligned}$$

This allows for the estimate

$$\begin{aligned}
|\sigma(\zeta, s)| &= (2\pi)^{-3/2} \frac{1}{|\zeta|} \left| \int_0^\infty \mathrm{e}^{-\zeta r} \left(\mu'(r) \frac{\sin(rs)}{s} + \mu(r) \cos rs \right) dr \right| \\
&\leq (2\pi)^{-3/2} \frac{1}{|\zeta|} \int_0^{4R} \mathrm{e}^{-\nu r} \left(\frac{C_\mu}{R} r + 1 \right) dr \\
&\leq 4R (2\pi)^{-3/2} \frac{1}{|\zeta|} \left(4 C_\mu E_1(4\nu R) + E_0(4R\nu) \right)
\end{aligned}$$

with

$$E_1(t) = \frac{1 - \mathrm{e}^{-t}(1+t)}{t^2} \le E_0^2(t).$$

Hence,

$$|\sigma(\zeta, s)| \le 4R(2\pi)^{-3/2} \frac{E_0(4R\nu)}{|\zeta|} (1 + 4C_\mu E_0(4R\nu)).$$

Since $E_0(t) \le 1$, we end up with

$$|\sigma(\zeta, s)| \le 4R(1 + 4C_\mu)(2\pi)^{-3/2} \frac{E_0(4R\nu)}{|\zeta|}.$$

As a consequence, we have proved that

$$\begin{aligned} |\zeta| \, \|z_\mu\| &\le 4R(1 + 4C_\mu) E_0(4R\nu) \|\phi\|, \\ \|\partial_i z_\mu\| &\le 4R E_0(4R\nu) \|\phi\|, \end{aligned}$$

so that we have

$$\|z_\mu\|_{|\zeta|} \le \sqrt{2 + (1 + 4C_\mu)^2} \, (16\pi R) \, E_0(4R\nu) \|\phi\|.$$

This shows (5.3). In the following we estimate higher order derivatives. For the product $s^2\sigma(s)$, we get

$$\begin{aligned} \left|s^2 \sigma(\zeta, s)\right| &= (2\pi)^{-3/2} \left| \int_0^\infty \mathrm{e}^{-\zeta r} \mu(r) \, \partial_r \cos(rs) \, dr \right| \\ &\le (2\pi)^{-3/2} \left(\left| \int_0^\infty \cos(rs) \, \partial_r \left(\mathrm{e}^{-\zeta r} \mu(r) \right) dr \right| + 1 \right) \\ &\le (2\pi)^{-3/2} |\zeta| \left| \int_0^\infty \cos(rs) \, \mathrm{e}^{-\zeta r} \mu(r) \, dr \right| \\ &\quad + (2\pi)^{-3/2} \left(\left| \int_0^\infty \cos(rs) \, \mathrm{e}^{-\zeta r} \mu'(r) \, dr \right| + 1 \right) \\ &=: T^{\mathrm{I}} + T^{\mathrm{II}}. \end{aligned}$$

The estimates

$$T^{\mathrm{I}} \le (2\pi)^{-3/2} 4R E_0(4R\nu) |\zeta|, \qquad T^{\mathrm{II}} \le (2\pi)^{-3/2} 4C E_0(4R\nu) \tag{5.10}$$

follow from the properties of μ (cf. (5.2)). As a simple consequence we obtain for $m \ge 2$

$$\left|s^2 \sigma(\zeta, s)\right| \le (2\pi)^{-3/2} 4(C + R|\zeta|) E_0(4R\nu) \tag{5.11}$$

and

$$\sup_{0<s<\lambda|k|} |s^m \sigma(\zeta, s)| \le (2\pi)^{-3/2} 4C_0 \left(\frac{C + R|\zeta|}{1 + 4R\nu} \right) (\lambda |k|)^{m-2}. \tag{5.12}$$

Hence for $\alpha \in \mathbb{N}_0^3$, $|\alpha| = 2$, we have

$$\|\partial^\alpha z_\mu\| \le 4(R|\zeta| + C) E_0(4R\nu) \|\phi\|$$

and

$$\|\nabla^p z_{\mathcal{A}}\| = \sqrt{\sum_{\substack{\alpha\in\mathbb{N}_0^3\\|\alpha|=p}} \frac{p!}{\alpha!}\,\|\partial^\alpha z_{\mathcal{A}}\|^2} \leq C' E_0\,(4R\nu)\,(1+|\zeta|)\,(\lambda\,|k|)^{p-2}\,3^{p/2}\,\|\phi\|$$

$$\leq C''\frac{1+|\zeta|}{1+\nu}\left(\sqrt{3}\lambda\,|k|\right)^{p-2}\|\phi\| \qquad \forall p\in\mathbb{N}_{\geq 2}, \tag{5.13}$$

where $\alpha! := \alpha_1!\alpha_2!\alpha_3!$. The bounds (5.13) express the desired estimate (5.5).

A direct application of (5.11) does not lead to (5.4) as it introduces an undesired factor $|\zeta|$. This is removed by noting that it suffices to consider $s = \|\xi\|$ with $s \geq \lambda|k|$ and that only the estimates for T^I need to be refined. This is achieved with an integration by parts:

$$\begin{aligned}\left|T^{\mathrm{I}}\right| &= (2\pi)^{-3/2}|\zeta|\left|\left(\frac{\zeta}{\zeta^2+s^2}+\int_0^{4R}\frac{\mathrm{e}^{-\zeta r}(\zeta\cos(rs)-s\sin(rs))}{\zeta^2+s^2}\mu'(r)\,dr\right)\right|\\ &\leq (2\pi)^{-3/2}\left(\frac{|\zeta|^2}{|\zeta^2+s^2|}\left(1+\frac{C}{R}\int_0^{4R}\mathrm{e}^{-\nu r}\,dr\right)\right.\\ &\qquad\left.+\frac{|\zeta|\,s}{|\zeta^2+s^2|}\left|\int_0^{4R}\mathrm{e}^{-\zeta r}\sin(rs)\,\mu'(r)\,dr\right|\right).\end{aligned}$$

Observe

$$\frac{|\zeta|^2}{|\zeta^2+s^2|} = \frac{|\zeta|^2}{\sqrt{\left(\nu^2+s^2-k^2\right)^2+4\nu^2k^2}} \leq \frac{|\zeta|^2}{s^2-k^2} \leq \left(\frac{|\zeta|}{\operatorname{Im}\zeta}\right)^2\frac{1}{\lambda^2-1}.$$

Also we have

$$\frac{s|\zeta|}{\nu^2+(s^2-k^2)} \leq \frac{\lambda\,|k|\,|\zeta|}{\nu^2+k^2\left(\lambda^2-1\right)} \leq \frac{\lambda}{\lambda^2-1}\frac{|\zeta|}{|\operatorname{Im}\zeta|}.$$

Hence,

$$\left|T^{\mathrm{I}}\right| \leq (2\pi)^{-3/2}\frac{C}{\lambda-1}\left(\frac{|\zeta|}{\operatorname{Im}\zeta}\right)^2. \tag{5.14}$$

This leads to

$$\left|s^2\sigma\left(\zeta,s\right)\right| \leq (2\pi)^{-3/2}C\frac{\lambda}{\lambda-1}\left(\frac{|\zeta|}{\operatorname{Im}\zeta}\right)^2 \quad \text{for } |s|\geq\lambda\,|k|$$

and, in turn,

$$\left|s^p\sigma\left(\zeta,s\right)\right| \leq (2\pi)^{-3/2}C\frac{\lambda}{\lambda-1}\left(\frac{|\zeta|}{\operatorname{Im}\zeta}\right)^2(\lambda\,|\operatorname{Im}\zeta|)^{p-2} \quad \text{for } |s|\geq\lambda\,|k|\,,\ p=0,1,2.$$

From this, assertion (5.4) follows. ∎

5.2. The Helmholtz solution with Robin boundary conditions. In this section, we will derive a regularity result for $\zeta \in D_\beta^c$ in the spirit of Lemma 5.1 for the interior problem with Robin boundary conditions:

$$-\Delta u + \zeta^2 u = f \quad \text{in } \Omega, \qquad \partial_n u + \zeta u = g \quad \text{on } \Gamma. \tag{5.15}$$

Note that $\Omega \in \mathcal{A}$ implies well-posedness of (5.15) via Lemma 4.2. The solution operator for (5.15) is denoted by $S_\zeta : L^2(\Omega) \times H^{1/2}(\Gamma) \to V$.

THEOREM 5.3. *Let $\Omega \in \mathcal{A}$ and fix $\beta > 0$. Then there exist constants C, $\gamma > 0$ such that for every $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$, and $\zeta \in D_\beta^c$, the solution $u = S_\zeta(f, g)$ of (5.15) can be written as $u = u_{\mathcal{A}} + u_{H^2}$, where, for all $p \in \mathbb{N}_0$,*

$$\|u_{\mathcal{A}}\|_{|\zeta|} \le C \left(\frac{1}{1+\operatorname{Re}(\zeta)} \|f\| + \frac{1}{\sqrt{1+\operatorname{Re}(\zeta)}} \frac{1}{\sqrt{|\zeta|}} \|g\|_{\Gamma,|\zeta|} \right), \tag{5.16}$$

$$\|\nabla^{p+2} u_{\mathcal{A}}\|_{L^2(\Omega)} \le C \frac{\gamma^p}{|\zeta|} \max\{p, |\zeta|\}^{p+2} \left(\frac{1}{1+\operatorname{Re}(\zeta)} \|f\| + \frac{1}{\sqrt{|\zeta|}} \|g\|_{\Gamma,|\zeta|} \right), \tag{5.17}$$

$$\|u_{H^2}\|_{H^2(\Omega)} + |\zeta| \|u_{H^2}\|_{|\zeta|} \le C \left(\|f\| + \|g\|_{\Gamma,|\zeta|} \right). \tag{5.18}$$

Proof. The proof is the generalization of the proof in [16] for real wavenumbers to more general $\zeta \in \mathbb{C}_{\ge 0}^{\circ}$ with emphasis on the explicit dependence of the estimates on the real and imaginary parts. It follows from Lemmas 5.11 and 5.12, which will be presented in section 5.3. ∎

5.3. The solution operators N_ζ, S_ζ^{Δ}, S_ζ^{L}, and S^ζ. For the analysis we introduce low- and high-pass frequency filters for a bounded domain as well as for its boundary. Let $E_\Omega : L^2(\Omega) \to L^2(\mathbb{R}^3)$ be the extension operator of Stein [24, Chap. VI]. Then for $f \in L^2(\Omega)$ we set

$$L_\Omega f := \left(L_{\mathbb{R}^d} (E_\Omega f) \right)|_\Omega \quad \text{and} \quad H_\Omega f := \left(H_{\mathbb{R}^d} (E_\Omega f) \right)|_\Omega \tag{5.19}$$

for $L_{\mathbb{R}^d}$ and $H_{\mathbb{R}^d}$ defined in (5.7), for some $\lambda > 1$ and $\zeta \in \mathbb{C}_{\ge 0}^{\circ}$, $\operatorname{Im}\zeta \ne 0$. By [16, Lemmas 4.2, 4.3], these operators have the following stability properties:

$$\|L_\Omega f\|_{H^s(\Omega)} \le C_s \|f\|_{H^s(\Omega)}, \qquad 0 \le s, \tag{5.20}$$

$$\|H_\Omega f\|_{H^{s'}(\Omega)} \le C_{s,s'} |\lambda \operatorname{Im}\zeta|^{s'-s} \|f\|_{H^s(\Omega)}, \qquad 0 \le s' \le s, \tag{5.21}$$

where the constant C_s depends on s and $C_{s,s'}$ depends on s, s' but is independent of λ and ζ.

To define frequency filters on the boundary we employ a lifting operator G^N defined in Lemma 5.4 below with the mapping property $G^N : H^s(\Gamma) \to H^{3/2+s}(\Omega)$ for every $s > 0$ and $\partial_n G^N g = g$. We then define H_Γ^N and L_Γ^N depending on $\lambda > 0$ via (5.7) by

$$H_\Gamma^N(g) := \partial_n H_\Omega \left(G^N g \right), \qquad L_\Gamma^N(g) := \partial_n L_\Omega \left(G^N g \right). \tag{5.22}$$

In particular, we have $H_\Gamma^N : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $L_\Gamma^N : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$.

LEMMA 5.4 (definition of lifting G^N). *Let $\Omega \in \mathcal{A}$. Given $\zeta \in \mathbb{C}_{\ge 0}^{\circ}$, define $u := G^N g$ as the solution of*

$$-\Delta u + |\zeta|^2 u = 0 \quad \text{in } \Omega, \qquad \partial_n u = g \quad \text{on } \partial\Omega.$$

Then the following hold:

$$\|G^N g\|_{|\zeta|} \lesssim \frac{1}{\sqrt{|\zeta|}} \|g\|_\Gamma, \tag{5.23}$$

$$\|G^N g\|_{H^2(\Omega)} \lesssim \|g\|_{\Gamma,|\zeta|}\,. \tag{5.24}$$

Proof. The energy estimate (5.23) follows from the coercivity of the pertinent sesquilinear form. The H^2-estimate follows from elliptic regularity theory. ∎

LEMMA 5.5 (properties of L_Γ and H_Γ). *Let $\Omega \in \mathcal{A}$ and $\zeta \in D_\beta^c$. Fix $q \in (0,1)$. Then there is $\lambda > 1$ in the definition of L_Γ^N and H_Γ^N (cf.* (5.7), (5.19), (5.22)*) such that the following hold (with implied constants independent of q):*

$$\|L_\Gamma^N g\|_{H^s(\Gamma)} \lesssim |\zeta|^{s-1/2} \|g\|_{\Gamma,|\zeta|}, \tag{5.25}$$

$$\|H_\Gamma^N g\|_{H^s(\Gamma)} \lesssim q^{1/2-s} |\zeta|^{s-1/2} \|g\|_{\Gamma,|\zeta|} \tag{5.26}$$

for $s \in \{0, 1/2\}$.

Proof. Recall that $L_\Gamma^N g := \gamma_0 g^N$, where

$$g^N := \langle n^*, \nabla L_\Omega G^N g\rangle \tag{5.27}$$

and n^* denotes an analytic extension of the normal $n : \Gamma \to \mathbb{S}_2$ to a tubular neighborhood $T \subset \Omega$ of Γ. The trace estimate (3.5) yields

$$\begin{aligned}
\|L_\Gamma^N g\|_\Gamma &\le C \frac{1}{\sqrt{|\zeta|}} \|g^N\|_{|\zeta|,T} \\
&= C \left(\sqrt{|\zeta|} \|g^N\|_{L^2(T)} + \frac{1}{\sqrt{|\zeta|}} \|\nabla g^N\|_{L^2(T)} \right) \\
&\le C \left(\sqrt{|\zeta|} \|\nabla L_\Omega G^N g\| + \frac{1}{\sqrt{|\zeta|}} \left\|\nabla \nabla^\intercal L_\Omega G^N g\right\| \right),
\end{aligned}$$

where $\nabla\nabla^\intercal$ denotes the Hessian of a function. From (5.20)

$$\|L_\Gamma^N g\|_\Gamma \lesssim \sqrt{|\zeta|} \|G^N g\|_{H^1(\Omega)} + \frac{1}{\sqrt{|\zeta|}} \left\|G^N g\right\|_{H^2(\Omega)} \overset{\text{Lemma 5.4}}{\lesssim} |\zeta|^{-1/2} \|g\|_{\Gamma,|\zeta|}.$$

For $s = 1/2$, we note that

$$\|L_\Gamma^N g\|_{H^{1/2}(\Gamma)} \lesssim \|G^N g\|_{H^2(\Omega)} \lesssim \|g\|_{\Gamma,|\zeta|}.$$

The proof of (5.26) is similar. We note that

$$\begin{aligned}
\|H_\Omega G^N\|_{H^2(\Omega)} &\overset{(5.21)}{\lesssim} \|G^N\|_{H^2(\Omega)} \lesssim \|g\|_{\Gamma,|\zeta|}, \\
\|H_\Omega G^N\|_{H^1(\Omega)} &\overset{(5.21)}{\lesssim} q\,|\zeta|^{-1} \|G^N\|_{H^2(\Omega)} \lesssim q\,|\zeta|^{-1} \|g\|_{\Gamma,|\zeta|},
\end{aligned}$$

where q is related to λ via (5.21) and can be made arbitrarily small by selecting λ appropriately. Hence, recalling that $H_\Gamma^N g = \partial_n H_\Omega G^N g$, we get

$$\begin{aligned}
\|H_\Gamma^N g\|_{H^{1/2}(\Gamma)} &\lesssim \|G^N g\|_{H^2(\Omega)} \lesssim \|g\|_{\Gamma,|\zeta|}, \\
\|H_\Gamma^N g\|_\Gamma &\lesssim \|G^N g\|_{H^1(\Omega)}^{1/2} \|G^N g\|_{H^2(\Omega)}^{1/2} \lesssim q^{1/2} |\zeta|^{-1/2} \|g\|_{\Gamma,|\zeta|}.
\end{aligned}$$

∎

Next, we introduce the solution operators N_ζ, S_ζ^Δ, S_ζ^L.

1. We denote by $u := N_\zeta f = G(\zeta) * f$ the solution of the full space Helmholtz problem with Sommerfeld radiation condition (in the weak sense):
$$(-\Delta + \zeta^2)u = f \text{ in } \mathbb{R}^3,$$
$$\left|\frac{\partial u}{\partial r} + \zeta u\right| = o\left(\|x\|^{-1}\right) \text{ as } \|x\| \to \infty,$$
for $f \in L^2(\mathbb{R}^3)$ with compact support. Here $\partial/\partial r$ denotes the derivative in radial direction $x/\|x\|$.
2. $S_\zeta^\Delta(g)$ is the solution operator to the problem
$$-\Delta u + |\zeta|^2 u = 0 \text{ in } \Omega,$$
$$\partial_n u + \zeta u = g \text{ on } \Gamma,$$
for $g \in L^2(\Gamma)$.
3. We define $S_\zeta^L(f, g) := S_\zeta(L_\Omega f, L_\Gamma^N g)$ as the solution operator to the problem (2.1) for analytic right-hand sides $L_\Omega f$, $L_\Gamma^N g$.

The proof of the next lemma is a consequence of the proof of Lemma 5.1.

LEMMA 5.6 (properties of N_ζ). *Let $\zeta \in \mathbb{C}_{\geq 0}^\circ$ and $\operatorname{Im}\zeta \neq 0$. For $f \in L^2(\mathbb{R}^3)$ with $\operatorname{supp} f \subset B_R := B_R(0)$, the function $u = N_\zeta f$ satisfies $-\Delta u + \zeta^2 u = f$ on B_R. There exists $C > 0$ depending only on R and μ as in* (5.2) *such that for any $\lambda > 1$ (appearing in the definition of the operator $H_{\mathbb{R}^3}$ defined in* (5.7)*) it holds that*

$$\|N_\zeta(H_{\mathbb{R}^3} f)\|_{|\zeta|, B_R} \leq C \frac{1}{\lambda - 1}\left(\frac{|\zeta|}{|\operatorname{Im}\zeta|}\right)^3 |\operatorname{Im}\zeta|^{-1} \|f\|_{L^2(\mathbb{R}^3)}, \tag{5.28a}$$

$$\|N_\zeta(H_{\mathbb{R}^3} f)\|_{H^2(B_R)} \leq C \frac{\lambda}{\lambda - 1}\left(\frac{|\zeta|}{\operatorname{Im}\zeta}\right)^2 \|f\|_{L^2(\mathbb{R}^3)}. \tag{5.28b}$$

Furthermore, for $\beta > 0$ there is $C_\beta > 0$ such that for any $q \in (0, 1)$ one can select $\lambda > 1$ such that for all $\zeta \in D_\beta^c$

$$\|N_\zeta(H_{\mathbb{R}^3} f)\|_{|\zeta|, B_R} \leq q |\operatorname{Im}\zeta|^{-1} \|f\|_{L^2(\mathbb{R}^3)}, \tag{5.29a}$$

$$\|N_\zeta(H_{\mathbb{R}^3} f)\|_{H^2(B_R)} \leq C_\beta \|f\|_{L^2(\mathbb{R}^3)}, \tag{5.29b}$$

where C_β is independent of λ.

Proof. Inequalities (5.28) are a direct consequence of the proof of Lemma 5.1. The bounds (5.29) follow from (5.28). ∎

The next two lemmas generalize the results in [16, Lemmas 4.5, 4.6] to complex wavenumbers ζ.

LEMMA 5.7 (properties of S_ζ^Δ). *Let Ω be a bounded Lipschitz domain and $\beta > 0$. For $g \in L^2(\Gamma)$ the function $u = S_\zeta^\Delta(g)$ satisfies*

$$\|u\|_{|\zeta|} \lesssim \|g\|_{H^{-1/2}(\Gamma)}, \quad \|u\|_{|\zeta|} \lesssim |\zeta|^{-1/2} \|g\|_\Gamma, \quad \|u\|_\Gamma \lesssim |\zeta|^{-1} \|g\|_\Gamma \tag{5.30}$$

uniformly for all $\zeta \in D_\beta^c$. If Γ is smooth and $g \in H^{1/2}(\Gamma)$, then additionally

$$\|u\|_{H^2(\Omega)} \lesssim \|g\|_{\Gamma, |\zeta|}.$$

Proof. The proof is a direct consequence of the proof of [16, Lemma 4.6]. ∎

A combination of Lemmas 5.5 and 5.7 implies the following corollary.

COROLLARY 5.8 (properties of $S_\zeta^\Delta \circ H_\Gamma^N$). *Let $\Omega \in \mathcal{A}$, $\beta > 0$, and let $q \in (0,1)$. There exists $\lambda > 1$ defining the high frequency filter H_Γ^N (cf. (5.7), (5.19), (5.22)) such that for every $g \in H^{1/2}(\Gamma)$ and every $\zeta \in D_\beta^c$ we have*

$$\|S_\zeta^\Delta(H_\Gamma^N g)\|_{|\zeta|} \leq q\frac{1}{|\zeta|}\|g\|_{\Gamma,|\zeta|},$$
$$\|S_\zeta^\Delta(H_\Gamma^N g)\|_{H^2(\Omega)} \lesssim \|g\|_{\Gamma,|\zeta|}.$$

LEMMA 5.9 (analyticity of S_ζ^L). *Let $\Omega \in \mathcal{A}$ and let $\lambda > 1$ appearing in the definition of L_Ω and L_Γ^N be fixed. Then there exist constants C, $\gamma > 0$ independent of $\zeta \in D_\beta^c$ such that, for every $g \in H^{1/2}(\Gamma)$ and $f \in L^2(\Omega)$, the function $u_\mathcal{A} = S_\zeta(L_\Omega f, L_\Gamma^N g)$ is analytic on Ω and satisfies for all $p \in \mathbb{N}_0$ the estimates*

$$\|u_\mathcal{A}\|_{|\zeta|} \leq C\left(\frac{1}{1+\operatorname{Re}(\zeta)}\|f\| + \frac{1}{\sqrt{1+\operatorname{Re}(\zeta)}}\frac{1}{\sqrt{|\zeta|}}\|g\|_{\Gamma,|\zeta|}\right), \tag{5.31}$$

$$\begin{aligned}\left\|\nabla^{p+2}u_\mathcal{A}\right\| &\leq C\gamma^p\max\{|\zeta|,p+2\}^{p+2}|\zeta|^{-1}\\ &\quad\times\left(\frac{1}{1+\operatorname{Re}(\zeta)}\|f\| + \frac{1}{\sqrt{1+\operatorname{Re}(\zeta)}}\frac{1}{\sqrt{|\zeta|}}\|g\|_{\Gamma,|\zeta|}\right).\end{aligned} \tag{5.32}$$

Proof. From Lemma 4.2, we have

$$\|u_\mathcal{A}\|_{|\zeta|} \leq C\left(\frac{1}{1+\operatorname{Re}(\zeta)}\|L_\Omega f\| + \frac{1}{\sqrt{1+\operatorname{Re}(\zeta)}}\|L_\Gamma^N g\|_\Gamma\right). \tag{5.33}$$

The combination of (5.33), Lemma 5.4, Lemma 5.5, and (5.20) leads to

$$\|u_\mathcal{A}\|_{|\zeta|} \leq C\left(\frac{1}{1+\operatorname{Re}(\zeta)}\|f\| + \frac{1}{\sqrt{1+\operatorname{Re}(\zeta)}}|\zeta|^{-1/2}\|g\|_{\Gamma,|\zeta|}\right).$$

To estimate higher derivatives, we employ [12, Prop. 5.4.5] in a similar way as in the proof of [16, Lemma 4.13]. To apply [12, Prop. 5.4.5] an estimate of the constant

$$C_{G_1} := |\zeta|^{-1}\sqrt{\|g^N\|_{L^2(T)}^2 + |\zeta|^{-2}\|\nabla g^N\|_{L^2(T)}^2}$$

is needed, where g^N is defined in (5.27). We track the dependence of C_{G_1} on $|\zeta|$ in a modified way (compared to [16, p. 1225]) and employ inequalities (5.23), (5.24) to obtain

$$C_{G_1} \leq C|\zeta|^{-2}\|g\|_{\Gamma,|\zeta|}. \tag{5.34}$$

Estimate (5.32) then follows from [12, Prop. 5.4.5]. ∎

COROLLARY 5.10. *Fix $\beta > 0$. Let f, $\tilde{f} \in L^2(\Omega)$ and $\zeta \in D_\beta^c$. Set $\tilde{u} = N_\zeta(H_\Omega\tilde{f})$. If g has the form $g = (\partial_n\tilde{u} + \zeta\tilde{u})$, then the function $u_\mathcal{A} = S_\zeta(L_\Omega f, L_\Gamma^N g)$ satisfies for all $p \in \mathbb{N}_0$*

$$\begin{aligned}\left\|\nabla^{p+2}u_\mathcal{A}\right\| &\leq C_\beta\gamma^p\max\{|\zeta|,p+2\}^{p+2}|\zeta|^{-1}\\ &\quad\times\left(\frac{1}{1+\operatorname{Re}(\zeta)}\|f\| + \frac{1}{\sqrt{1+\operatorname{Re}(\zeta)}}\frac{1}{\sqrt{|\zeta|}}\|\tilde{f}\|\right).\end{aligned}$$

If $\tilde{f} = f$, this gives

$$\left\|\nabla^{p+2} u_{\mathcal{A}}\right\| \leq C_\beta \gamma^p \max\{|\zeta|, p+2\}^{p+2} |\zeta|^{-1} \frac{1}{(1+\operatorname{Re}\zeta)} \|f\|.$$

Proof. We proceed in the same way as in [16, Lemma 4.12] with $k = \operatorname{Im}\zeta$ and estimate the constant C_{G_1} in (5.34). Lemma 5.6 and (3.5) lead to

$$\|\tilde{u}\|_\Gamma \leq C\,|\zeta|^{-1/2}\,\|\tilde{u}\|_{|\zeta|} \leq C\,|\zeta|^{-3/2}\,\|\tilde{f}\|, \tag{5.35a}$$

$$\|\tilde{u}\|_{H^{1/2}(\Gamma)} \leq C\|\tilde{u}\|_{H^1(\Omega)} \leq C\,|\zeta|^{-1}\,\|\tilde{f}\|, \tag{5.35b}$$

$$\|\partial_n \tilde{u}\|_\Gamma \leq C\,\|\nabla\tilde{u}\|^{1/2}\,\|\tilde{u}\|^{1/2}_{H^2(\Omega)} \leq C\,|\zeta|^{-1/2}\,\|\tilde{f}\|, \tag{5.35c}$$

$$\|\partial_n \tilde{u}\|_{H^{1/2}(\Gamma)} \leq C\,\|\tilde{u}\|_{H^2(\Omega)} \leq C\|\tilde{f}\|. \tag{5.35d}$$

This implies

$$\|\partial_n \tilde{u} + \zeta\tilde{u}\|_{L^2(\Gamma)} \lesssim \frac{1}{\sqrt{|\zeta|}}\|\tilde{f}\|, \qquad \|\partial_n \tilde{u} + \zeta\tilde{u}\|_{H^{1/2}(\Gamma)} \lesssim \|\tilde{f}\|, \tag{5.36}$$

and

$$C_{G_1} := \frac{1}{|\zeta|^{3/2}}\|\left(\partial_n\tilde{u} + \zeta\tilde{u}\right)\|_\Gamma + \frac{1}{|\zeta|^2}\|\left(\partial_n\tilde{u} + \zeta\tilde{u}\right)\|_{H^{1/2}(\Gamma)} \leq C\,|\zeta|^{-2}\,\|\tilde{f}\|.$$

In the same way as at the end of the proof of Lemma 5.9 we obtain

$$\begin{aligned}\left\|\nabla^{p+2} u_{\mathcal{A}}\right\| &\leq C_\beta \gamma^p \max\{|\zeta|, p+2\}^{p+2} |\zeta|^{-1} \\ &\quad\times\left(\frac{1}{1+\operatorname{Re}(\zeta)}\,\|f\| + \frac{1}{\sqrt{1+\operatorname{Re}(\zeta)}}\frac{1}{\sqrt{|\zeta|}}\|\tilde{f}\| + \frac{1}{|\zeta|}\|\tilde{f}\|\right). \qquad \square\end{aligned}$$

LEMMA 5.11 (properties of $S_\zeta(f,0)$). *Let $\beta > 0$, $\Omega \in \mathcal{A}$, and $\zeta \in D^c_\beta$. For every $q \in (0,1)$, there exist constants C, $K > 0$ depending on β such that for every $f \in L^2(\Omega)$ and $\zeta \in D^c_\beta$, the function $u = S_\zeta(f,0)$ can be written as $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$, where*

$$\begin{aligned}\|u_{\mathcal{A}}\|_{|\zeta|} &\leq \frac{C}{1+\operatorname{Re}(\zeta)}\|f\|,\\ \|\nabla^{p+2}u_{\mathcal{A}}\| &\leq \frac{C}{1+\operatorname{Re}(\zeta)}|\zeta|^{-1}K^p\max\{p+2,|\zeta|\}^{p+2}\|f\| \qquad \forall p \in \mathbb{N}_0,\\ \|u_{H^2}\|_{|\zeta|} &\leq q|\zeta|^{-1}\|f\|,\\ \|u_{H^2}\|_{H^2(\Omega)} &\leq C\|f\|.\end{aligned}$$

The remainder $\widetilde{u}$ satisfies

$$-\Delta\widetilde{u} + \zeta^2\widetilde{u} = \widetilde{f}, \qquad \partial_n\widetilde{u} + \zeta\widetilde{u} = 0$$

for a function $\widetilde{f} \in L^2(\Omega)$ with $\|\widetilde{f}\| \leq q\|f\|$.

Proof. Define $u^{\mathrm{I}}_{\mathcal{A}} := S_\zeta(L_\Omega f, 0), u^{\mathrm{I}}_{H^2} := N_\zeta(H_\Omega f)$. Here, the parameter λ defining the filter operators L_Ω and H_Ω is still at our disposal and will be selected at the end of the proof. Then, $u^{\mathrm{I}}_{\mathcal{A}}$ satisfies the desired bounds by Lemma 5.9. Lemma 5.6 gives

$$\|u^{\mathrm{I}}_{H^2}\|_{|\zeta|} \leq q'|\zeta|^{-1}\|f\| \qquad \text{and} \qquad \|u^{\mathrm{I}}_{H^2}\|_{H^2(\Omega)} \leq C\|f\|.$$

Also, the parameter $q' \in (0,1)$ depends on λ and is still at our disposal. In fact, in view of the statement of Lemma 5.6 it can be made sufficiently small by taking λ sufficiently large.

The function $u^{\mathrm{I}} := u - (u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{I}}_{H^2})$ solves

$$-\Delta u^{\mathrm{I}} + \zeta^2 u^{\mathrm{I}} = 0, \qquad \partial_n u^{\mathrm{I}} + \zeta u^{\mathrm{I}} = -\left(\partial_n u^{\mathrm{I}}_{H^2} + \zeta u^{\mathrm{I}}_{H^2}\right). \tag{5.37}$$

Next, we define the functions $u^{\mathrm{II}}_{\mathcal{A}}$ and $u^{\mathrm{II}}_{H^2}$ by

$$u^{\mathrm{II}}_{\mathcal{A}} := S_\zeta\left(0, -L^N_\Gamma\left(\partial_n u^{\mathrm{I}}_{H^2} + \zeta u^{\mathrm{I}}_{H^2}\right)\right), \qquad u^{\mathrm{II}}_{H^2} := S^\Delta_\zeta\left(-H^N_\Gamma\left(\partial_n u^{\mathrm{I}}_{H^2} + \zeta u^{\mathrm{I}}_{H^2}\right)\right).$$

Then, the analytic part $u^{\mathrm{II}}_{\mathcal{A}}$ satisfies again the desired analyticity bounds by Lemma 5.9 and Corollary 5.10 . For the function $u^{\mathrm{II}}_{H^2}$ we obtain from Corollary 5.8 and inequalities (5.35) (set $\tilde{u} = u^{\mathrm{I}}_{H^2}$) the estimates

$$\begin{aligned}
\|u^{\mathrm{II}}_{H^2}\|_{|\zeta|} &\le q'|\zeta|^{-1}\left\|\partial_n u^{\mathrm{I}}_{H^2} + \zeta u^{\mathrm{I}}_{H^2}\right\|_{\Gamma,|\zeta|} \le Cq'|\zeta|^{-1}\|f\|,\\
\|u^{\mathrm{II}}_{H^2}\|_{H^2(\Omega)} &\lesssim \left\|\partial_n u^{\mathrm{I}}_{H^2} + \zeta u^{\mathrm{I}}_{H^2}\right\|_{\Gamma,|\zeta|} \lesssim \|f\|.
\end{aligned}$$

Recall $\nu = \operatorname{Re}\zeta$ and $k = -\operatorname{Im}\zeta$. We now set $u_{\mathcal{A}} := u^{\mathrm{I}}_{\mathcal{A}} + u^{\mathrm{II}}_{\mathcal{A}}$ and $u_{H^2} := u^{\mathrm{I}}_{H^2} + u^{\mathrm{II}}_{H^2}$ and conclude that the function $\widetilde{u} := u - (u_{\mathcal{A}} + u_{H^2})$ satisfies

$$-\Delta\widetilde{u} + \zeta^2\widetilde{u} = \widetilde{f} := 2\left(k^2 + \mathrm{i}\,\nu k\right)u^{\mathrm{II}}_{H^2}, \qquad \partial_n\widetilde{u} + \zeta\widetilde{u} = 0.$$

For $\widetilde{f}$ we obtain

$$\|\widetilde{f}\| \le 2|\zeta|\|u^{\mathrm{II}}_{H^2}\|_{|\zeta|} \le 2Cq'\|f\|.$$

Hence, by taking λ sufficiently large so that q' is sufficiently small, we arrive at the desired bound. ∎

LEMMA 5.12 (properties of $S_\zeta(0,g)$). *Let $\beta > 0$ and $\Omega \in \mathcal{A}$. Let $q \in (0,1)$. Then there exist constants C, $K > 0$ independent of $\zeta \in D^c_\beta$ (but depending on β) such that for every $g \in H^{1/2}(\Gamma)$ the function $u = S_\zeta(0,g)$ can be written as $u = u_{\mathcal{A}} + u_{H^2} + \widetilde{u}$, where for all $p \in \mathbb{N}_0$*

$$\begin{aligned}
\|u_{\mathcal{A}}\|_{|\zeta|} &\le \frac{C}{\sqrt{1+\operatorname{Re}\zeta}}\frac{1}{\sqrt{|\zeta|}}\|g\|_{\Gamma,|\zeta|},\\
\|\nabla^{p+2}u_{\mathcal{A}}\| &\le C|\zeta|^{-1}K^p\max\{p+2,|\zeta|\}^{p+2}\frac{1}{\sqrt{1+\operatorname{Re}(\zeta)}}\frac{1}{\sqrt{|\zeta|}}\|g\|_{\Gamma,|\zeta|},\\
\|u_{H^2}\|_{|\zeta|} &\le q\frac{1}{|\zeta|}\|g\|_{\Gamma,|\zeta|},\\
\|u_{H^2}\|_{H^2(\Omega)} &\le C\|g\|_{\Gamma,|\zeta|}.
\end{aligned}$$

The remainder $\widetilde{u}$ satisfies

$$-\Delta\widetilde{u} + \zeta^2\widetilde{u} = 0, \qquad \partial_n\widetilde{u} + \zeta\widetilde{u} = \widetilde{g}$$

for some $\widetilde{g} \in H^{1/2}(\Gamma)$ with $\|\widetilde{g}\|_{\Gamma,|\zeta|} \le q\|g\|_{\Gamma,|\zeta|}$.

Proof. The proof is very similar to that of Lemma 5.11. Define

$$u^{\mathrm{I}}_{\mathcal{A}} := S_\zeta(0, L^N_\Gamma g) \qquad \text{and} \qquad u^{\mathrm{I}}_{H^2} := S^\Delta_\zeta(H^N_\Gamma g).$$

Then $u_{\mathcal{A}}^{\mathrm{I}}$ is analytic and satisfies the desired analyticity estimates by Lemma 5.9. For $u_{H^2}^{\mathrm{I}}$ we have by Corollary 5.8

$$\|u_{H^2}^{\mathrm{I}}\|_{|\zeta|} \leq q' \frac{1}{|\zeta|} \|g\|_{\Gamma,|\zeta|}, \qquad \|u_{H^2}^{\mathrm{I}}\|_{H^2(\Omega)} \lesssim \|g\|_{\Gamma,|\zeta|}, \tag{5.38}$$

where $q' \in (0,1)$ is at our disposal and depends on the parameter λ in the definition of H_Γ^N and L_Γ^N. Recalling $\nu = \operatorname{Re}\zeta$ and $k = -\operatorname{Im}\zeta$ the function $u^{\mathrm{I}} := u_{\mathcal{A}}^{\mathrm{I}} + u_{H^2}^{\mathrm{I}}$ satisfies

$$-\Delta u^{\mathrm{I}} + \zeta u^{\mathrm{I}} = -2 \underbrace{\left(k^2 + \mathrm{i}\,\nu k\right)}_{=\mathrm{i}\,k\zeta} u_{H^2}^{\mathrm{I}}, \qquad \partial_n u^{\mathrm{I}} + \zeta u^{\mathrm{I}} = g$$

together with

$$\|2\,\mathrm{i}\,k\zeta u_{H^2}^{\mathrm{I}}\| \leq 2|\zeta| \|u_{H^2}^{\mathrm{I}}\|_{|\zeta|} \overset{(5.38)}{\leq} 2q' \|g\|_{\Gamma,|\zeta|}. \tag{5.39}$$

Next, we define $u_{\mathcal{A}}^{\mathrm{II}}$ and $u_{H^2}^{\mathrm{II}}$ by

$$u_{\mathcal{A}}^{\mathrm{II}} := S_\zeta \left(L_\Omega \left(2\left(k^2 + \mathrm{i}\,\nu k\right) u_{H^2}^{\mathrm{I}}\right), 0\right) \text{ and } u_{H^2}^{\mathrm{II}} := N_\zeta \left(H_\Omega \left(2\left(k^2 + \mathrm{i}\,\nu k\right) u_{H^2}^{\mathrm{I}}\right)\right).$$

Here, in order to apply the operator N_ζ, we extend $H_\Omega\left(2\left(k^2 + \mathrm{i}\,\nu k\right) u_{H^2}^{\mathrm{I}}\right)$ by zero outside of Ω. By Lemma 5.9 and (5.39), we see that $u_{\mathcal{A}}^{\mathrm{II}}$ satisfies the desired analyticity estimates. For the function $u_{H^2}^{\mathrm{II}}$, we obtain from Lemma 5.6

$$\begin{aligned}
\|u_{H^2}^{\mathrm{II}}\|_{|\zeta|} &\leq q' |\zeta|^{-1} \|2\,\mathrm{i}\,k\zeta u_{H^2}^{\mathrm{I}}\| \overset{(5.38)}{\leq} 2\left(q'\right)^2 |\zeta|^{-1} \|g\|_{\Gamma,|\zeta|}, \\
\|u_{H^2}^{\mathrm{II}}\|_{H^2(\Omega)} &\leq C_\beta \||\zeta|^2 u_{H^2}^{\mathrm{I}}\| \leq C_\beta |\zeta| \|u_{H^2}^{\mathrm{I}}\|_{|\zeta|} \overset{(5.38)}{\leq} C_\beta q' \|g\|_{\Gamma,|\zeta|}.
\end{aligned}$$

We set $u_{\mathcal{A}} := u_{\mathcal{A}}^{\mathrm{I}} + u_{\mathcal{A}}^{\mathrm{II}}$ and $u_{H^2} := u_{H^2}^{\mathrm{I}} + u_{H^2}^{\mathrm{II}}$. Then $u_{\mathcal{A}}$ and u_{H^2} satisfy the desired estimates and $\widetilde{u} := u - (u_{\mathcal{A}} + u_{H^2})$ satisfies

$$-\Delta \widetilde{u} + \zeta^2 \widetilde{u} = 0, \qquad \partial_n \widetilde{u} + \zeta \widetilde{u} = \widetilde{g} := -\left(\partial_n u_{H^2}^{\mathrm{II}} + \zeta u_{H^2}^{\mathrm{II}}\right)$$

with

$$\begin{aligned}
\|\widetilde{g}\|_{\Gamma,|\zeta|} &\lesssim |\zeta|^{3/2} \|u_{H^2}\|_\Gamma + |\zeta|^{1/2} \|\partial_n u_{H^2}\|_\Gamma + |\zeta| \|u_{H^2}^{\mathrm{II}}\|_{H^{1/2}(\Gamma)} + \left\|\partial_n u_{H^2}^{\mathrm{II}}\right\|_{H^{1/2}(\Gamma)} \\
&\leq C' \left(|\zeta| \|u_{H^2}^{\mathrm{II}}\|_{|\zeta|} + \left\|u_{H^2}^{\mathrm{II}}\right\|_{H^2(\Omega)}\right) \leq C'' q' \|g\|_{\Gamma,|\zeta|}.
\end{aligned}$$

The result follows by selecting λ sufficiently large so that q' is sufficiently small. ∎

6. Discretization. We apply the regularity theory of the previous section to the *hp*-finite element method (*hp*-FEM). We assume throughout this section that $\Omega \in \mathcal{A}$.

Definition 6.1 (adjoint approximability). *Let $\tilde{S}_\zeta(u) = \overline{S_\zeta(\overline{u}, 0)}$ be the solution operator of the adjoint problem: find $z \in V$ such that*

$$a_{\overline{\zeta}}(z, w) = (u, w) \quad \forall w \in V. \tag{6.1}$$

Let $S \subset V$ be a closed subspace. We define the adjoint approximability by

$$\eta(S) := \sup_{f \in L^2(\Omega)\backslash\{0\}} \inf_{v \in S} \frac{\|\tilde{S}_\zeta f - v\|_{|\zeta|}}{\|f\|}.$$

6.1. Discrete inf-sup constant $\gamma_{\mathbf{disc}}$ and quasi-optimality. For $\operatorname{Re}\zeta > 0$, the existence and uniqueness of the Galerkin solution follow from Lemma 4.1. If $\zeta = -\mathrm{i}\,k$ is purely imaginary, well-posedness and quasi-optimality of the Galerkin discretization are shown in [16] under the restriction that

$$|k|\,\eta(S) \leq \frac{1}{4(1+C_b)},$$

where C_b is the constant appearing in (3.3). In the next theorem, we derive an estimate of the discrete inf-sup constant for general $\zeta \in \mathbb{C}_{\geq 0}^{\circ}$.

THEOREM 6.2. *Let $\Omega \in \mathcal{A}$. For $\zeta \in \mathbb{C}_{\geq 0}$ let the sesquilinear form a_ζ be given by* (2.3)*. Then the discrete inf-sup constant*

$$\gamma_{\text{disc}} := \inf_{u \in S\backslash\{0\}} \sup_{v \in S\backslash\{0\}} \frac{|a_\zeta\,(u,v)|}{\|u\|_{|\zeta|}\,\|v\|_{|\zeta|}}$$

satisfies the following:

1. *If* $\operatorname{Re}\zeta > 0$*, then*

$$\gamma_{\text{disc}} \geq \frac{\operatorname{Re}\zeta}{|\zeta|}.$$

2. *If* $\zeta \in \mathbb{C}_{\geq 0}^{\circ}$ *and* $\frac{(\operatorname{Im}\zeta)^2}{|\zeta|}\eta(S) \leq \frac{1}{4(1+C_b)}$*, then*

$$\gamma_{\text{disc}} \geq c\frac{1+\operatorname{Re}\zeta}{|\zeta|} \tag{6.2}$$

for a constant c independent of ζ.

Remark 6.3. The resolution condition (6.2) is not an artifact of the theory: in [13, Ex. 3.7], a domain Ω, a finite element space S, and a purely imaginary wavenumber $\zeta = -\mathrm{i}\,k$ are presented where the Galerkin discretization leads to a system matrix that is not invertible.

Proof of Theorem 6.2. Let $\zeta = \nu - \mathrm{i}\,k$. The first statement follows directly from the continuous inf-sup constant in Lemma 4.1. We prove the second statement. Let $u \in S$ and choose $v = u + z$, where $z = 2k^2\tilde{S}_\zeta\,(u)$ (cf. Definition 6.1). Then it is simple to check that

$$\operatorname{Re} a(u, u+z) \geq \|u\|_{|\zeta|}^2.$$

Let $z_S \in S$ be the best approximation of z with respect to the $\|\cdot\|_{|\zeta|}$ norm. Then

$$\begin{aligned}
\operatorname{Re} a(u, u+z_S) &= \operatorname{Re} a(u, u+z) + \operatorname{Re} a(u, z_S - z) \\
&\geq \|u\|_{|\zeta|}^2 - (1+C_b)\|u\|_{|\zeta|}\|z - z_S\|_{|\zeta|} \\
&\geq \|u\|_{|\zeta|}^2 - 2k^2(1+C_b)\eta(S)\|u\|_{|\zeta|}\|u\| \\
&\geq \left(1 - 2\frac{k^2}{|\zeta|}(1+C_b)\eta(S)\right)\|u\|_{|\zeta|}^2 \\
&\geq \frac{1}{2}\|u\|_{|\zeta|}^2.
\end{aligned}$$

Moreover, we employ $\|z - z_S\|_{|\zeta|} \le \|z\|_{|\zeta|}$ and Lemma 4.2 to get

$$\begin{aligned}\|u + z_S\|_{|\zeta|} &\le \|u\|_{|\zeta|} + \|z - z_S\|_{|\zeta|} + \|z\|_{|\zeta|}\\ &\le \left(1 + C_{\mathrm{stab}} \frac{4k^2}{(1+\nu)|\zeta|}\right) \|u\|_{|\zeta|},\end{aligned}$$

and, in turn, we have proved that

$$\gamma_{\mathrm{disc}} \ge \frac{\operatorname{Re} a(u, u + z_S)}{\|u\|_{|\zeta|}\|u + z_S\|_{|\zeta|}} \ge \frac{1}{2}\frac{1}{1 + \frac{4k}{|\zeta|}\frac{k}{\nu+1}C_{\mathrm{stab}}}. \tag{6.3}$$

A simple calculation shows that there exists a constant $c > 0$ independent of $\zeta \in \mathbb{C}^{\circ}_{\ge 0}$ such that the right-hand side in (6.3) is bounded from below by the right-hand side in (6.2). ∎

THEOREM 6.4. *Let $\Omega \in \mathcal{A}$. Assume that $\operatorname{Re}\zeta > 0$. Then the Galerkin method based on S is quasi-optimal; i.e., for every $u \in V$ there exists a unique $u_S \in S$ with $a(u - u_S, v) - b(u - u_S, v) = 0$ for all $v \in S$, and*

$$\|u - u_S\|_{|\zeta|} \le \frac{|\zeta|}{\operatorname{Re}(\zeta)}(1 + C_b) \inf_{v\in S} \|u - v\|_{|\zeta|}, \tag{6.4}$$

$$\|u - u_S\|_{L^2(\Omega)} \le (1 + C_b)\eta(S)\|u - u_S\|_{|\zeta|}. \tag{6.5}$$

Inequality (6.4) is a direct consequence of the discrete inf-sup constant proved in Theorem 6.2. Estimate (6.5) follows from the proof of the next theorem (see (6.9)). We note here that for $\zeta \in D_\beta$, the ratio $|\zeta|/\operatorname{Re}\zeta$ is bounded from above, and no resolution assumption is required. In the next theorem, we find that under a resolution assumption, the estimate (6.4) can be improved such that it is nondegenerate for $\operatorname{Re}\zeta \longrightarrow 0$.

THEOREM 6.5. *Let $\Omega \in \mathcal{A}$. If*

$$\operatorname{Re}\zeta \ge 0 \qquad \text{and} \qquad \frac{(\operatorname{Im}\zeta)^2}{|\zeta|}\eta(S) \le \frac{1}{4(1 + C_b)}, \tag{6.6}$$

then the Galerkin method based on S is quasi-optimal and

$$\|u - u_S\|_{|\zeta|} \le 2(1 + C_b) \inf_{v\in S} \|u - v\|_{|\zeta|}, \tag{6.7}$$

$$\|u - u_S\|_{L^2(\Omega)} \le (1 + C_b)\eta(S)\|u - u_S\|_{|\zeta|}. \tag{6.8}$$

Proof. Let $e := u - u_S$ and define $\psi := \tilde{S}_\zeta e$. Let ψ_S be the best approximation to ψ with respect to the $\|\cdot\|_{|\zeta|}$-norm. The Galerkin orthogonality implies

$$\begin{aligned}\|e\|^2 = a_\zeta(e, \psi) = a_\zeta(e, \psi - \psi_S) &\le (1 + C_b)\|e\|_{|\zeta|}\|\psi - \psi_S\|_{|\zeta|}\\ &\le (1 + C_b)\eta(S)\|e\|_{|\zeta|}\|e\|.\end{aligned}$$

This yields

$$\|e\| \le (1 + C_b)\eta(S)\|e\|_{|\zeta|} \tag{6.9}$$

in both cases. Let $k = -\operatorname{Im}\zeta$. We compute for $v \in S$

$$\begin{aligned}\|e\|^2_{|\zeta|} &\le \operatorname{Re}\left(a_\zeta(e, e) + 2k^2\|e\|^2\right) \le \operatorname{Re}\left(a_\zeta(e, u - v) + 2k^2\|e\|^2\right)\\ &\le (1 + C_b)\|e\|_{|\zeta|}\|u - v\|_\zeta + 2\frac{k^2}{|\zeta|}(1 + C_b)\eta(S)\|e\|^2_{|\zeta|},\end{aligned}$$

which leads to (6.7) under the condition $\frac{k^2}{|\zeta|}\eta(S) \le \frac{1}{4(1+C_b)}$. ∎

6.2. Impact on *hp*-FEM approximation. We have shown in section 6.1 that the Galerkin solution $u_S \in S$ of the Helmholtz problem with Robin boundary conditions (5.15) with $\zeta \in D_\beta^c$ is quasi-optimal for any closed subspace $S \subset V$ if the adjoint approximability $\eta(S)$ fulfills the resolution condition

$$\frac{(\operatorname{Im}\zeta)^2}{|\zeta|}\eta(S) \le \frac{1}{4(1+C_b)}.$$

Let S be the hp-FEM space described in [15, sect. 5]. For the the sake of completeness, we repeat the main definitions and assumption.

Definition 6.6 (triangulation (according to the setting in [4])). *The triangulation $\mathcal{T}_h$ consists of images K under the element maps $F_K : \hat{K} \to K$, where $\hat{K}$ is the reference tetrahedron. Element maps of elements which share an edge or a face induce the same parametrization on that edge or face. We denote by h the maximal mesh width, i.e., $h := max_{K\in\mathcal{T}_h} \operatorname{diam} K$.*

We emphasize that curved elements are allowed in Definition 6.6 while hanging nodes are not. Next, we impose an assumption on the element maps F_K.

Assumption 6.7 (quasi-uniform regular triangulation). *Each element map F_K can be written as a composition $F_K = R_K \circ A_K$ of an affine map A_K and an analytic map R_K. The maps R_K and A_K satisfy for constants $C_A, C_R, \gamma > 0$ independent of h*

$$\|A_K'\|_{L^\infty(\hat{K})} \le C_A h, \qquad \|(A_K')^{-1}\|_{L^\infty(\hat{K})} \le C_A h^{-1},$$
$$\|(R_K')^{-1}\|_{L^\infty(A_K(\hat{K}))} \le C_R h^{-1}, \qquad \|\nabla^n R_K\|_{L^\infty(A_K(\hat{K}))} \le C_R \gamma^n n!.$$

For a triangulation $\mathcal{T}_h$ defined as in Definition 6.6 with element maps F_K satisfying Assumption 6.7 we define the space of piecewise mapped polynomials by

$$S^{p,1}(\mathcal{T}_h) := \left\{u \in H^1(\Omega) \,|\, \forall K \in \mathcal{T}_h : u_{|_K} \circ F_K \text{ is a polynomial of degree } p\right\}.$$

The following theorem holds.

Theorem 6.8. *Let $\Omega \in \mathcal{A}$ and $\beta > 0$. There exist constants C, $\sigma > 0$ that are independent of $\zeta \in D_\beta^c$ such that for every $f \in L^2(\Omega)$ the function $u = \tilde{S}_\zeta(f) = \overline{S_\zeta(\overline{f}, 0)}$ satisfies for the regular decomposition $u = u_\mathcal{A} + u_{\mathcal{H}^2}$ given by Theorem* 5.3

$$\frac{|\operatorname{Im}\zeta|^2}{|\zeta|}\inf_{w\in S}\|u_{H^2} - w\|_{|\zeta|} \le C\frac{|\operatorname{Im}\zeta|}{|\zeta|}\left(\frac{|\zeta|h}{p} + \left(\frac{|\zeta|h}{p}\right)^2\right)\|f\|, \tag{6.10a}$$

$$\begin{aligned}\frac{|\operatorname{Im}\zeta|^2}{|\zeta|}&\inf_{w\in S}\|u_\mathcal{A} - w\|_{|\zeta|} \\ &\le C\frac{|\operatorname{Im}\zeta|^2}{|\zeta|}\frac{1}{1+\operatorname{Re}(\zeta)}\left(\frac{1}{p} + \frac{|\zeta|h}{\sigma p}\right)\left(\frac{h}{p} + \left(\frac{|\zeta|h}{\sigma p}\right)^p\right)\|f\|.\end{aligned} \tag{6.10b}$$

Proof. See [15, sect. 5] and in particular the proof of [15, Thm. 5.5] for details. ∎

Similarly as in [14], one can show that for the Galerkin method based on $S := S^{p,1}(\mathcal{T}_h)$ and fixed constant $C > 0$, there exists a constant $c > 0$ such that the resolution condition

$$R(c, C, \zeta): \qquad \frac{|\zeta|h}{p} \le c \quad \text{and} \quad p \ge C\log\left(\mathrm{e} + \frac{|\operatorname{Im}(\zeta)|}{1+\operatorname{Re}(\zeta)}\right)$$

ensures convergence and quasi-optimality. Indeed, for every $f \in L^2(\Omega)$, we can apply the splitting to $\tilde{S}_\zeta(f) =: u = u_{\mathcal{A}} + u_{H^2}$. Then

$$\begin{aligned}\frac{(\operatorname{Im}\zeta)^2}{|\zeta|}\eta(S) &= \frac{(\operatorname{Im}\zeta)^2}{|\zeta|} \sup_{f\in L^2(\Omega)\setminus\{0\}} \inf_{w\in S} \frac{\|\tilde{S}_\zeta f - w\|_{|\zeta|}}{\|f\|} \\ &\le \frac{(\operatorname{Im}\zeta)^2}{|\zeta|} \sup_{f\in L^2(\Omega)\setminus\{0\}} \left(\inf_{w\in S} \frac{\|u_{\mathcal{A}} - w\|_{|\zeta|}}{\|f\|} + \inf_{w\in S} \frac{\|u_{H^2} - w\|_{|\zeta|}}{\|f\|} \right).\end{aligned}$$

The estimates (6.10) show that for sufficiently small c the resolution condition $R(c, C, \zeta)$ implies the resolution assumption (6.6) and therefore quasi-optimality and optimal convergence for the Galerkin solution.

If $\zeta \in D_\beta$, no resolution condition is needed for the quasi-optimality of the problem (cf. Theorem 6.4). In that case, the solution is typically smooth in the domain and exhibits, for large $\operatorname{Re}\zeta$, a boundary layer. Such problems can be handled by suitable meshes capable of resolving the layers such as Shishkin meshes in the context of the h-version of the FEM [12, 17, 23] and "spectral boundary layer meshes" in the context of the *hp*-FEM [12, 22].

7. Numerical experiments. We consider the domain $\Omega = B_1(0) \subset \mathbb{R}^2$ and the equation

$$\begin{aligned} -\Delta u + \zeta^2 u &= 1 \qquad \text{in } \Omega, \\ \partial_n u + \zeta u &= 0 \qquad \text{on } \Gamma = \partial\Omega. \end{aligned}$$

In terms of the Bessel functions J_0, J_1 and polar coordinates, the solution is given as

$$u(r) = c_1 J_0(\mathrm{i}\,\zeta r) + \zeta^{-2}, \qquad c_1 = \frac{\mathrm{i}}{\zeta^2}\,\frac{1}{J_1(\mathrm{i}\,\zeta) - \mathrm{i}\,J_0(\mathrm{i}\,\zeta)}.$$

We consider values of ζ with $\zeta = |\zeta| e^{\mathrm{i}\,\alpha}$, where

$$\begin{aligned} &\alpha = \frac{\pi}{2}(1 - \widetilde{\alpha}), \qquad \widetilde{\alpha} \in \{0, 2^{-6}, 2^{-4}, 2^{-2}, 2^{-1}, 1\}, \\ &|\zeta| \in \{1, 10, 50, 100\}. \end{aligned}$$

The purely imaginary wavenumber corresponds to the choice $\alpha = \pi/2$ and $\alpha = 0$ to the real-valued case. We consider the h-FEM on quasi-uniform meshes for $p \in \{1, 2, 3, 4\}$. The results are presented Figure 1, where the *error* is plotted versus the number of degrees of freedom per wavelength

$$N_{|\zeta|} = \frac{2\pi\sqrt{DOF}}{|\zeta|\sqrt{|\Omega|}} = \mathcal{O}\left(\frac{p}{h|\zeta|}\right).$$

The calculations were carried out within the *hp*-FEM framework NgSolve [20, 21]. The following features are visible in Figure 1:

(a) A plateau before convergence starts.

(b) A pollution effect for ζ close to the imaginary axis ($\alpha = \pi/2$). That is, for $\operatorname{Arg}\zeta$ close to $\pi/2$, the asymptotic quasi-optimality starts for larger $N_{|\zeta|}$ as $|\zeta|$ becomes larger.

(c) The pollution effect decreases with increasing polynomial degree. In particular, the asymptotic behavior is reached for smaller values of $N_{|\zeta|}$ as p is increased.

(d) The pollution effect decreases with decreasing angle α, i.e., as ζ enters the sectorial zone towards the real axes.

The main focus of our paper is the development of a detailed analysis for two parameter regimes: if the resolution condition is satisfied, the convergence is quasi-optimal; if not, the constants grow outside the sectorial case as $\operatorname{Re}\zeta \to 0$, and even well-posedness of the Galerkin discretization is not guaranteed. A stability and convergence analysis for certain preasymptotic regimes is available in the literature; see, e.g., [5, 7, 25].

The observation (a) reflects a natural resolution condition for the problem class under consideration; that is, the best approximation error can only be expected to be small if $N_{|\zeta|} \sim |\zeta|h/p$ is small. The pollution effect observed in (b) is well documented for the purely imaginary case $\operatorname{Re}\zeta = 0$. Figure 1 shows that it is present also for $\operatorname{Re}\zeta \neq 0$ (and large $\operatorname{Im}\zeta$), albeit in a mitigated form. Theorem 6.5 quantifies how this pollution effect is weakened as the ratio $\operatorname{Re}\zeta/\operatorname{Im}\zeta$ increases. More specifically, the resolution condition (6.2), which results from applying Theorem 6.5 to high order methods, illustrates the helpful effect of $\operatorname{Re}\zeta \neq 0$. In the limiting case $\operatorname{Im}\zeta = 0$, the Galerkin method is an energy projection method, and even monotone convergence can be expected in the energy norm on sequences of nested meshes.

The observation (c) is also well documented for the purely imaginary case $\operatorname{Re}\zeta = 0$ and mathematically explained in [15, 16]. The regularity of the present work permits us to extend the *hp*-FEM analysis of [15, 16] to the case $\operatorname{Re}\zeta \neq 0$ as done in section 6.2. The observation that the asymptotic convergence regime is reached for smaller $N_{|\zeta|}$ as p is increased can be understood qualitatively from Theorem 6.5 and the bounds (6.10) for η. Consider, for notational simplicity, the case $\operatorname{Re}\zeta = 0$. Then quasi-optimality of the *hp*-FEM is reached if

$$|\zeta|\eta(S) \lesssim \left(1 + \frac{h|\zeta|}{p}\right)\left(\frac{h|\zeta|}{p} + |\zeta|\left(\frac{h|\zeta|}{\sigma p}\right)^p\right) \overset{!}{\lesssim} 1.$$

Recalling $N_{|\zeta|} = O(h|\zeta|/p)$ allows us to simplify the condition for quasi-optimality as

$$\frac{1}{N_{|\zeta|}} + |\zeta|\left(\frac{1}{\sigma N_{|\zeta|}}\right)^p \overset{!}{\lesssim} 1.$$

This shows that for larger p quasi-optimality of the *hp*-FEM may be expected for small $N_{|\zeta|}$.

Finally, observation (d) can again be explained by Theorem 6.5 since the factor $(\operatorname{Im}\zeta)^2/|\zeta|$ is reduced as the ratio $\operatorname{Re}\zeta/\operatorname{Im}\zeta$ increases.

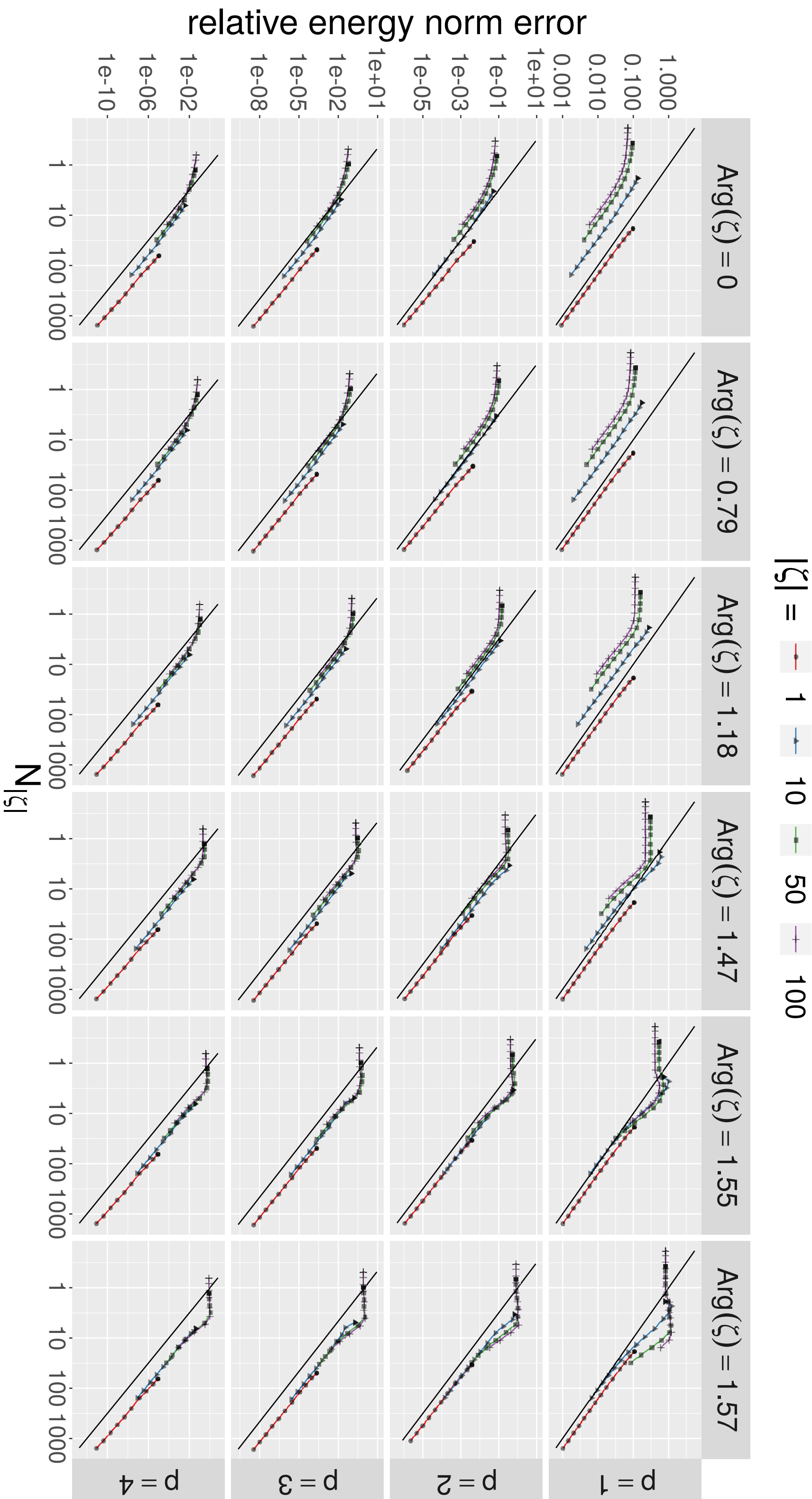

FIG. 1. *Plots with relative weighted* H^1*-norm for* $\zeta = |\zeta| \exp(\mathrm{i} \operatorname{Arg} \zeta)$, $|\zeta| \in \{1, 10, 50, 100\}$, $\operatorname{Arg} \zeta = \frac{\pi}{2}(1-\widetilde{\alpha})$, *for* $\widetilde{\alpha} \in \{0, 2^{-6}, 2^{-4}, 2^{-2}, 2^{-1}, 1\}$, $p \in \{1, 2, 3, 4\}$ *and different "numbers of degrees of freedom per wavelength"* $N_{|\zeta|}$.

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